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The Additive Completion of kth Powers

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Let $k \geqslant 2$ be a fixed integer. For positive integers $M \leqslant N$, let $S_k(M, N)$ denote the set of all sets $A \subset [0, M]$ such that, for all positive integers $n \leqslant N$, n can be written as $n = a + b^k$ with $a \in A$ and b a positive integer. Define $f_k(M, N) = \min\{|A|: A \in S_k(M, N)\}$. Given $\varepsilon > 0$, we prove that there exists a $\delta > 0$ such that for all sufficiently large N

$$f_k(\delta N, N) \geqslant (k - \varepsilon)\, N^{1 - 1/k}.$$

1. INTRODUCTION AND RESULTS

Suppose $k \geqslant 2$ is a fixed integer. For positive integers $M \leqslant N$, let $S_k(M, N)$ denote the set of all sets $A \subset [0, M]$ such that every positive integer $n \leqslant N$ can be written as $n = a + b^k$ with $a \in A$ and b a positive integer. Define $f_k(M, N) = \min\{|A|: A \in S_k(M, N)\}$. Many authors considered the problem of obtaining a lower bound for $f_k(M, N)$ in the case $M = N$ (see [1–6]. The latest result is due to J. Cilleruedo [4], who showed that

$$f_k(N, N) \geqslant N^{1 - 1/k} \left\{ \frac{1}{\Gamma(2 - 1/k)\, \Gamma(1 + 1/k)} + o(1) \right\}. \tag{1}$$

The aim of this paper is to estimate $f_k(M, N)$ when M is small compared to N. Let $M_k = M_k(N)$ denote the smallest integer M such that $S_k(M, N)$ is non-empty. For example, if S is an integer such that $S \geqslant 4k$, then $M_k(S^k - 1) = S^k - (S-1)^k - 1$ and $M_k(S^k + 1) = (S+1)^k - S^k - 1$. In general, we have the following proposition.

PROPOSITION. *Let $B = [N^{1/k}]$. Then*

$$B^k - (B-1)^k - 1 \leqslant M_k \leqslant (B+1)^k - B^k - 1. \tag{2}$$

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Our main result is the following

THEOREM 1. *Let $k \geqslant 2$ be an integer. There exist constants $\varepsilon_0 = \varepsilon_0(k) > 0$ and $N_0 = N_0(k) > 1$ such that, if $N \geqslant N_0$ and $3k^2 N^{-1/2k} \leqslant \varepsilon \leqslant \varepsilon_0$, then we have*

$$f_k(\delta N, N) \geqslant (k - \varepsilon)\, N^{1-1/k},$$

where $\delta = \delta(\varepsilon) = \varepsilon^2/(9k^3)$.

The referee kindly suggested the following question: What can be said about the upper bound of $f_k(M, N)$? This question can be answered by

THEOREM 2. *For all integers $k \geqslant 2$ and $M_k(N) \leqslant M \leqslant N$ we have*

$$f_k(M, N) \leqslant (B+1)^k - B^k, \tag{3}$$

where $B = [N^{1/k}]$.

From Theorem 1 and Theorem 2 we immediately get

COROLLARY. *Let $k \geqslant 2$ be an integer. There exist constants $0 < \delta_k < 1$ and $N_k > 1$ such that, if $N \geqslant N_k$ and $M_k/N < \delta < \delta_k$, then*

$$(k - 3k^{3/2}\delta^{1/2})\, N^{1-1/k} \leqslant f_k(\delta N, N) \leqslant k(1 + kN^{-1/k})\, N^{1-1/k}. \tag{4}$$

In particular, if $\delta = \delta(N)$ satisfies $\delta(N) \to 0$ as $N \to \infty$, then

$$f_k(\delta N, N) = (k + o(1))\, N^{1-1/k} \qquad (N \to \infty). \tag{5}$$

It may be conjectured that the asymptotic formula (5) holds uniformly for $M_k \leqslant M \leqslant N$. However, our method works only when $M = o(N)$.
For large k, we have the following result.

THEOREM 3. *Suppose $0 < \delta < 1$ is a fixed real number. There exist constants $k_0 = k_0(\delta) > 2$ and $N_k(\delta) > 2$ such that if $N \geqslant N_k(\delta)$ and $k \geqslant k_0$, then*

$$f_k(\delta N, N) \geqslant C(\delta)\, \frac{k^{\Delta}}{\log k} \left(1 - \frac{\log(1 + 1/\delta)}{\log k}\right) N^{1-1/k},$$

where

$$C(\delta) = \frac{\delta \Delta^{\Delta} \log(1 + 1/\delta)}{(1+\Delta)^{1+\Delta}}, \qquad \Delta = \frac{\log 1/\delta}{\log(1 + 1/\delta)}.$$

Remark. Our theorems also hold if b^k is replaced by $b^k+P_{k-1}(b)$ or $[b^c]$, where $P_{k-1}(b)$ is a polynomial of degree $k-1$ and $c\geqslant 2$ is any fixed real number.

2. PROOF OF THE PROPOSITION

The proof is short and simple. Let $A_1=\{0, 1, 2, ..., (B+1)^k-B^k-1\}$, where $B=[N^{1/k}]$. For each $n\leqslant N$, there exists a positive integer b such that $b^k\leqslant n\leqslant (b+1)^k$. Put $a=n-b^k$. Then $n=a+b^k$ and $a\in A_1$. This implies that $S_k((B+1)^k-B^k-1, N)$ is non-empty. Thus $M_k(N)\leqslant (B+1)^k-B^k-1$.

Now let A_2 denote a set in $S_k(M_k, N)$. Consider the number $n=B^k-1$. It is obvious that if we want to write n in the form $a+b^k$ with $a\in A_2$, we must have $b\leqslant B-1$ and $a\geqslant B^k-(B-1)^k-1$. Thus $M_k(N)\geqslant B^k-(B-1)^k-1$.

This completes the proof of the proposition.

3. PROOF OF THEOREM 1

Suppose that ε satisfies the condition of Theorem 1 and let $\delta=\varepsilon^2/(9k^3)$. Let $A\in S_k(\delta N, N)$. We consider the identity

$$\sum_{t\in A} x^t \sum_{b^k\leqslant N} x^{b^k}=\sum_{n\leqslant L} c(n)\, x^n, \tag{6}$$

where $L=\max\{n \mid n=a+b^k,\ a\in A,\ b^k\leqslant N\}$ and $c(n)=\#\{n=a+b^k \mid a\in A,\ b^k\leqslant N\}$. By the definition of $S_k(M, N)$ we have $c(n)\geqslant 1$ for all $n\leqslant N$ and $N\leqslant L\leqslant N+\delta N$.

Put $F(x)=\sum_{t\in A} x^t$, $G(x)=\sum_{b^k\leqslant N} x^{b^k}$ and $H(x)=\sum_{n\leqslant L} c(n)\, x^n$. Let $d=[2(1-1/k)/\varepsilon]$. By the Leibniz formula we get that

$$H^{(d)}(x)=\sum_{j=0}^{d} \binom{d}{j} F^{(j)}(x)\, G^{(d-j)}(x).$$

Setting $x=1$, we obtain

$$H^{(d)}(1)=\sum_{j=0}^{d} \binom{d}{j} F^{(j)}(1)\, G^{(d-j)}(1). \tag{7}$$

We now evaluate $H^{(d)}(1)$ and $F^{(j)}(1)$, $G^{(j)}(1)$ for $0 \leqslant j \leqslant d$. Obviously we have

$$H^{(d)}(1) = \sum_{d \leqslant n \leqslant L} c(n)\, n(n-1) \cdots (n-d+1)$$
$$\geqslant \sum_{d \leqslant n \leqslant N} n(n-1) \cdots (n-d+1). \tag{8}$$

where we used the fact $c(n) \geqslant 1$ for all $n \leqslant N$.

Let $y = y(x) = \sum_{n=0}^{N} x^n$. Then $(x-1)\, y = x^{N+1} - 1$. Using the Leibniz formula again we get

$$(x-1)\, y^{(d+1)}(x) + (d+1)\, y^{(d)}(x)$$
$$= (N+1)\, N(N-1) \cdots (N-d+1)\, x^{N-d}.$$

Setting $x = 1$ we obtain

$$y^{(d)}(1) = \frac{(N+1)\, N(N-1) \cdots (N-d+1)}{d+1}. \tag{9}$$

On the other hand,

$$y^{(d)}(1) = \sum_{d \leqslant n \leqslant N} n(n-1) \cdots (n-d+1). \tag{10}$$

Thus

$$H^{(d)}(1) \geqslant \frac{(N+1)\, N(N-1) \cdots (N-d+1)}{d+1}$$
$$> \frac{N^{d+1}}{d+1}\, e^{\sum_{j=1}^{d-1} \log(1 - j/N)}$$
$$= \frac{N^{d+1}}{d+1}\, e^{\sum_{j=1}^{d-1} -j/N + O(d^3/N^2)}$$
$$= \frac{N^{d+1}}{d+1}\, e^{-(d(d-1)/2N) + O(d^3/N^2)}$$
$$> \frac{N^{d+1}}{d+1} \left(1 - \frac{d(d-1)}{2N} + O(d^3/N^2)\right). \tag{11}$$

To estimate $F^{(j)}(1)$ we observe that

$$F^{(0)}(1) = \sum_{t \in A} 1 = |A| \tag{12}$$

and

$$F^{(j)}(1) = \sum_{t \in A,\, t \geqslant j} t(t-1) \cdots (t-j+1) \leqslant |A|\, \delta^j N^j, \qquad 1 \leqslant j \leqslant d. \quad (13)$$

Finally, we note that

$$G^{(0)}(1) = \sum_{b^k \leqslant N} 1 \leqslant N^{1/k} + 1 \quad (14)$$

and for $1 \leqslant j \leqslant d$

$$\begin{aligned} G^{(j)}(1) &= \sum_{j \leqslant b^k \leqslant N} b^k (b^k - 1) \cdots (b^k - j + 1) \\ &\leqslant \sum_{b^k \leqslant N} b^{jk} \leqslant \frac{N^{j+1/k}}{jk+1} + N^j. \end{aligned} \quad (15)$$

From (12)–(15) we obtain

$$\begin{aligned} H^{(d)}(1) &= \sum_{j=0}^{d} \binom{d}{j} F^{(j)}(1)\, G^{(d-j)}(1) \\ &\leqslant \sum_{j=0}^{d} \binom{d}{j} |A|\, \delta^j N^j \left(\frac{N^{d-j+1/k}}{(d-j)\, k+1} + N^{d-j} \right) \\ &\leqslant \frac{|A|\, N^{d+1/k}}{dk+1} \Bigg(1 + (1+\delta)^d\, (dk+1)\, N^{-1/k} \\ &\qquad + (dk+1) \sum_{j=1}^{d} \binom{d}{j} \delta^j \frac{1}{(d-j)\, k+1} \Bigg) \\ &\leqslant \frac{|A|\, N^{d+1/k}}{dk+1} \Bigg(1 + 2(dk+1)\, N^{-1/k} \\ &\qquad + (dk+1) \sum_{j=1}^{d} \binom{d}{j} \delta^j \frac{1}{(d-j)\, k+1} \Bigg), \end{aligned} \quad (16)$$

since

$$(1+\delta)^d < e^{d \log(1+\delta)} < e^{d\delta} < e^{1/2} < 2.$$

We now estimate the sum

$$\Sigma = \sum_{j=1}^{d} \binom{d}{j} \delta^j \frac{1}{(d-j)\, k+1}\, .$$

Write

$$\Sigma = \frac{d\delta}{(d-1)\,k+1} + \frac{d(d-1)\,\delta^2}{(d-2)\,k+1} + \sum_{3 \leqslant j \leqslant d/2} + \sum_{j > d/2},$$

say. We have

$$\sum_{j > d/2} < \delta^{d/2} \sum_{j > d/2} \binom{d}{j} < (2\delta^{1/2})^d < \frac{d\delta^2}{2k} \tag{17}$$

and

$$\begin{aligned} \sum_{3 \leqslant j \leqslant d/2} &< \frac{2}{dk} \sum_{3 \leqslant j \leqslant d} \binom{d}{j} \delta^j \\ &\leqslant \frac{2}{dk}\,((1+\delta)^d - 1 - d\delta) = \frac{2}{dk}\,(e^{d\log(1+\delta)} - 1 - d\delta) \\ &< \frac{2}{dk}\,(e^{d\delta} - 1 - d\delta) < \frac{2d\delta^2}{k}, \end{aligned} \tag{18}$$

where we have used the inequalities $\log(1+x) < x$ $(x > 0)$ and $e^x < 1 + x + x^2$ $(0 < x < 1/2)$.

Thus,

$$\Sigma < \frac{d\delta}{(d-1)\,k+1} + \frac{4d\delta^2}{k}, \tag{19}$$

if we notice

$$\frac{d(d-1)\,\delta^2}{(d-2)\,k+1} < \frac{3d\delta^2}{2k}.$$

Inserting (19) into (16) we get

$$H^{(d)}(1) < \frac{|A|\,N^{d+1/k}}{dk+1}\,\Delta_0 \tag{20}$$

with

$$\Delta_0 = 1 + 2(dk+1)\,N^{-1/k} + \frac{d(dk+1)\,\delta}{(d-1)\,k+1} + \frac{4d(dk+1)\,\delta^2}{k}.$$

Now from (11) and (20) we get

$$|A| > \Delta_1 N^{1-1/k}, \tag{21}$$

where

$$\begin{aligned}\Delta_1 &= \frac{dk+1}{d+1}\left(1-\frac{d(d-1)}{2N}+O\left(\frac{d^3}{N^2}\right)\right)\Delta_0^{-1}\\
&= k\left(1-\frac{1-1/k}{d+1}\right)\left(1-\frac{d(d-1)}{2N}+O\left(\frac{d^3}{N^2}\right)\right)\Delta_0^{-1}\\
&> k\left(1-\frac{1-1/k}{d+1}\right)\left(1-\frac{d(d-1)}{2N}+O\left(\frac{d^3}{N^2}\right)\right)\\
&\quad\times\left(1-2(dk+1)\,N^{-1/k}-\frac{d(dk+1)\,\delta}{(d-1)\,k+1}-\frac{4d(dk+1)\,\delta^2}{k}\right)\\
&\geqslant k-\varepsilon,\end{aligned}$$

if we recall the conditions on ε and the definitions of d and δ, and estimate Δ_0^{-1} using the inequality $1/(1+x)>1-x(|x|<1)$.

This completes the proof of Theorem 1.

4. PROOF OF THEOREM 2 AND COROLLARY

Without loss of generality, we may suppose $M\geqslant(B+1)^k-B^k-1$; otherwise Theorem 2 and the corollary follow from the proposition.

Let $A_0\in S_k(M,N)$ be such that $|A_0|=f_k(M,N)$. Since

$$\{0, 1, 2, ..., (B+1)^k-B^k-1\}\in S_k(M,N),$$

we have

$$|A_0|\leqslant(B+1)^k-B^k.$$

Thus (3) holds.

Now suppose $\delta_k>0$ is a sufficiently small positive constant and suppose δ satisfies the conditions of Theorem 2. Set $\varepsilon=3k^{3/2}\delta^{1/2}$. By Theorem 1 we have

$$|A_0|\geqslant(k-\varepsilon)\,N^{1-1/k} \tag{22}$$

if $M\leqslant\delta N$.

As for the upper bound we have

$$\begin{aligned}(B+1)^k-B^k&=k\int_B^{B+1} u^{k-1}\,du\\&=kB^{k-1}+k(k-1)\int_B^{B+1} du\int_B^u v^{k-2}\,dv\\&\leqslant kB^{k-1}+k(k-1)(B+1)^{k-2}\int_B^{B+1} du\int_B^u dv\\&\leqslant kB^{B+1}+k(k-1)(B+1)^{k-2}\leqslant kB^{k-1}+(kB)^2\\&\leqslant kN^{1-1/k}+k^2N^{1-2/k}.\end{aligned}\tag{23}$$

Now the corollary follows from (22) and (23).

5. PROOF OF THEOREM 3

Suppose $0<\delta<1$ is a fixed real number not depending on k and $A\in S_k(M, N)$. As in the proof of Theorem 1 we get

$$\begin{aligned}&\frac{(N+1)\,N(N-1)\cdots(N-d+1)}{d+1}\\&\qquad\leqslant |A|\,N^{d+1/k}\sum_{j=0}^{d}\binom{d}{j}\delta^j\frac{1}{(d-j)\,k+1}.\end{aligned}\tag{24}$$

We have

$$\begin{aligned}\sum_{j=0}^{d}\binom{d}{j}\delta^j\frac{1}{(d-j)\,k+1}&\leqslant\delta^d+k^{-1}\sum_{j=1}^{d}\binom{d}{j}\delta^j\\&\leqslant\delta^d+k^{-1}(1+\delta)^d\\&=\delta^d(1+k^{-1}(1+1/\delta)^d),\end{aligned}\tag{25}$$

provided $k>2(1+1/\delta)^d$.

Thus we get

$$|A|\,N^{1/k-1}\geqslant\frac{\delta^{-d}}{d+1}(1-k^{-1}(1+1/\delta)^d)\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{d-1}{N}\right).\tag{26}$$

Now take $d=[(\log k/D)/\log(1+1/\delta)]$, $D=1+\varDelta/\varDelta$, where $\varDelta$ is defined in Section 1. Thus Theorem 3 follows from (26). It should be noted that the choice of D is best possible.

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